Ramanujan's lost notebook and Ramanujan's circular summation

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Ramanujan's lost notebook contains Ramanujan's famous lost notebook and unpublished manuscripts in the Oxford library. In the spring of 1976, G.E. Andrews visited Trinity College Library at Cambridge University who found Ramanujan's lost notebook from the estate of G.N. Watson.

On page 54 in the Ramanujan's lost notebook, Ramanujan recorded the following claim (without proof) which is now well known as Ramanujan's circular summation.

For each positive integer n and $|ab| < 1$,

$$\sum_{-n/2<r\leq n/2}\left(\sum_{\substack{k=-\infty\\ k\equiv r(\mathrm{mod}\ n)}}^{\infty} a^{k(k+1)/(2n)}b^{k(k-1)/(2n)}\right)^n = f(a,b)F_n(ab),$$

where

$$F_n(q) = 1 + 2nq^{(n-1)/2} + \cdots, \qquad n \geq 3,$$

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, \quad |ab| < 1.$$

The function $f(a,b)$ is called Ramanujan's theta function.

We here introduce the investigation on Ramanujan's circular summation. The appellation circular summation was initiated by Seung Hwan Son.